ALGEBRA

1. PERMUTATION AND COMBINATION

Factorials

- The **Factorial** of a number is the multiplication of that number by every smaller number down to 1.
- The **Factorial Notation** is **n!**, where **n** represents the number and the "!" indicates the factorial process.
- Note the following: By definition 0! = 1
- Example: 8! = 8 * 7 *6 * 5 * 4 * 3 * 2 *1 = 40,320

Permutations & Combinations

A **combination** is an arrangement of items in which **ORDER DOES NOT MATTER.**

A **permutation** is an arrangement of items in a particular order.

Notice, **ORDER MATTERS**!

Example 1:

Which is the permutation, which is the combination problem?

a. You have six colors to choose from and you wish to choose three for a flag. How many choices of colors are possible? Combination

b. A flag with stripes of three different colors can use any one of six colors. How many flags are possible?

Permutation

c. A shopkeeper is rearranging a display of 8 vases in the shop window. How many different displays can she make?

Permutation

Permutations and Combinations

Number of permutations (order matters) of *n* things taken *r* at a time:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Number of combinations (order does not matter) of *n* things taken *r* at a time:

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

Number of different permutations of *n* objects where there are n_1 repeated items, n_2 repeated items, ... n_k repeated items

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Permutation Formulas (Order Matters)

1.) Number of permutations of n objects - no repetition allowed

$$n! = n \times (n-1) \times (n-2) \times (n-3) \ldots \times 2 \times 1$$

2.) Number of Permutations of r objects from n objects - repetition allowed

$$n^r$$

3.) Number of permutations of r objects from n objects - no repetition allowed

$$_nP_r = \frac{n!}{(n-r)!}$$

Permutations and Combinations

Permutation: (Order is important!)

Find $_{10}P_6$

$$_{10}P_6 = \frac{10!}{(10-6)!}$$

$$_{10}P_6 = \frac{10!}{4!} = \frac{10*9*8*7*6*5*4*3*2*1}{4*3*2*1}$$

$$_{10}P_6 = 10*9*8*7*6*5 \quad \text{or} \quad 151{,}200$$

There are 151,200 permutations of 10 objects taken 6 at a time.

$$\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$$

$$\Rightarrow \frac{1}{6!}+\frac{1}{7\times 6!}=\frac{x}{8\times 7\times 6!}$$

$$\Rightarrow \frac{1}{6!}\left(1+\frac{1}{7}\right)=\frac{x}{8\times 7\times 6!}$$

$$\Rightarrow 1+\frac{1}{7}=\frac{x}{8\times 7}$$

$$\Rightarrow \frac{8}{7}=\frac{x}{8\times 7}$$

$$\Rightarrow x=\frac{8\times 8\times 7}{7}$$

$$\therefore x=64$$

There are 30 people. A handshake needs 2 people.

This simply means in how many ways 2 people can be selected out of 30.

So the answer is ${}^{30}C_2$

$${}^{n}C_r = \frac{n!}{r!(n-r)!}$$

$$\therefore {}^{30}C_2 = \frac{30!}{2!(30-2)!} = \frac{30 \times 29}{2} = 435 = \textbf{Number of handshakes}$$

Tip:

If there are n people and they shake hands only once with each other, then,

Number of handshakes $= {}^{n}C_2 = \frac{n(n-1)}{2}$

11.1C Finding the Number of Permutations….. with Repeats

1. How many arrangements are there of four letters from the word *PREACHING*?

$${}_9P_4 \quad = \frac{9!}{(9-4)!} \quad = \frac{9!}{5!} \quad = 3024$$

2. How many distinct arrangements of *BRAINS* are there keeping the vowels together?

5! x 2! = 240

3. There are 6 different flags available for signaling. A signal consists of at least 4 flags tied one above the other. How many different signals can be made?

$${}_6P_4 + {}_6P_5 + {}_6P_6 = 1800$$

Combination of n things taken r at a time (p. 623)

Use the combination formula ${}_nC_r = n!/[(n-r)!r!]$ to answer these combination problems

1. If there are 20 people on a committee, how many ways could we pick a subcommittee of 7?

2 If there are 100 senators, how many ways could we pick a subcommittee of 7 of them?

3 If there are 72 potential jurors, how many different ways could they pick a jury of 12?

Example 3: Find the value of ${}^{12}P_2$, ${}^{9}P_1$ and ${}^{8}P_8$.

Solution. (i) $${}^{12}P_2 = \frac{12!}{10!} = \frac{12\times 11\times 10!}{10!} = 132$$ (ii) $${}^{9}P_1 = \frac{9!}{8!} = \frac{9\times 8!}{8!} = 9$$ (iii) $${}^{8}P_8 = \frac{8!}{0!} = 8!$$

Example 4: Find the value of ${}^{8}C_2$, ${}^{24}C_{22}$ and ${}^{36}C_{36}$.

Solution. (i) $${}^{8}C_2 = \frac{8!}{2!6!} = \frac{8\times 7}{2} = 28$$

(ii) $${}^{24}C_{22} = {}^{24}C_{(24-20)} = {}^{24}C_2 = \frac{24!}{2!22!} = \frac{24\times 23\times 22!}{2\times 1\times 22!} = 276 \qquad \left[\because\ {}^{n}C_r = {}^{n}C_{n-r}\right]$$

(iii) $${}^{36}C_{36} = {}^{36}C_0 = \frac{36!}{36!0!} = 36$$

1. Using all the letters of the word GIFT how many distinct words can be formed?

A. 22 words B. 24 words C. 256 words D. 200 words

2. Find out how many distinct three-digit numbers can be formed using all the digits of 1, 2, and 3.

A. 4 B. 5 C. 6 D. 7

3. In how many different ways can five friends sit for a photograph of five chairs in a row?

A. 120 ways B. 24 ways C. 240 ways D. 720 ways

4. In how many different ways can the letters of the word MAGIC can be formed?

A. 24 ways B. 120 ways C. 240 ways D. 720 ways

5. For the above word how many different types of arrangement are possible so that the vowels are always together?

A. 44 words B. 24 words C. 48 words D. 60 words

Answer:

1. B 24
2. C 6
3. A 120 ways
4. B 120 ways
5. C 48 words

SUMS FOR PRACTICE

- If ${}^{n}C_{r} - 1 = 36$, ${}^{n}C_{r} = 84$ and ${}^{n}C_{r+1} = 126$, then find ${}^{r}C_{2}$.
- If ${}^{n}C_{12} = {}^{n}C_{8}$, then n is equal to=
- In how many ways can a football team of 11 players be selected from 16 players? How many of them will (i) include 2 particular players? (ii) exclude 2 particular players?

ALGEBRA

2. BINOMIAL THEOREM

The Binomial Theorem

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + {}^nC_3\, a^{n-3} b^3 + \ldots + {}^nC_n\, a^0 b^n$$

$$= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots + \binom{n}{n} a^0 b^n$$

$$= a^n + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + {}^nC_3\, a^{n-3} b^3 + \ldots + b^n$$

Properties of binomial expansion $(a+b)^n$

$$\boxed{(a+b)^n = {}^nc_0 a^n b^0 + {}^nc_1 a^{n-1} b^1 + {}^nc_2 a^{n-2} b^2 + \ldots + {}^nc_{n-1} a^1 b^{n-1} + {}^nc_n a^0 b^n}$$

- Based on the binomial properties, the binomial theorem states that the following **binomial formula** is valid for all positive integer values of *n*.
- $(a+b)^n$ has n+1 terms as $0 \le k \le n$.
- The first term is a^n and the final term is b^n
- Progressing from the first term to the last, the exponent of *a* decreases by from term to term (start at **n and go down)** while the exponent of *b* increases by (start at **0 and go up)**.
- The sum of the exponents of *a* and *b* in each term is *n*.
- If the coefficient of each term is multiplied by the exponent of *a* in that term, and the product is divided by the number of that term, we obtain the coefficient of the next term.
- Coefficients of terms equidistant from beginning and end is same as ${}^nc_k = {}^nc_{n-k}$

Binomial expansion

Previously in the course we found that, when n is a positive whole number,

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n}$$

This is a finite series with $n + 1$ terms.

If n is negative or fractional then, provided that $|x| < 1$, the *infinite* series

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

will converge towards $(1 + x)^n$.

Example 1 Expand $\left(x^2 + \frac{3}{x}\right)^4, x \neq 0$

Solution By using binomial theorem, we have

$$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0(x^2)^4 + {}^4C_1(x^2)^3\left(\frac{3}{x}\right) + {}^4C_2(x^2)^2\left(\frac{3}{x}\right)^2 + {}^4C_3(x^2)\left(\frac{3}{x}\right)^3 + {}^4C_4\left(\frac{3}{x}\right)^4$$

$$= x^8 + 4.x^6 . \frac{3}{x} + 6.x^4 . \frac{9}{x^2} + 4.x^2 . \frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Expand $\left(a-\frac{b}{2}\right)^5$.

$$\left(a-\frac{b}{2}\right)^5$$

$$= a^5 + \frac{5!}{1!4!}a^4\left(-\frac{b}{2}\right) + \frac{5!}{2!3!}a^3\left(-\frac{b}{2}\right)^2 + \frac{5!}{3!2!}a^2\left(-\frac{b}{2}\right)^3$$

$$+ \frac{5!}{4!1!}a\left(-\frac{b}{2}\right)^4 + \left(-\frac{b}{2}\right)^5$$

$$= a^5 + 5a^4\left(-\frac{b}{2}\right) + 10a^3\left(\frac{b^2}{4}\right) + 10a^2\left(-\frac{b^3}{8}\right)$$

$$+ 5a\left(\frac{b^4}{16}\right) + \left(-\frac{b^5}{32}\right)$$

Notice that signs alternate positive and negative.

$$= a^5 - \frac{5}{2}a^4b + \frac{5}{2}a^3b^2 - \frac{5}{4}a^2b^3 + \frac{5}{16}ab^4 - \frac{1}{32}b^5$$

NC

$$(a+b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$(a-b)^6 = {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6$$

$$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

$$\therefore (a+b)^6 - (a-b)^6 = 2\left[6a^5b + 20a^3b^3 + 6ab^5\right]$$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6 = 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right) + 20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3 + 6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$$

$$= 2\left[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}\right]$$

$$= 2 \times 198\sqrt{6}$$

$$= 396\sqrt{6}$$

General Term of Binomial Expansion $(a+b)^n$

$$\text{Term } T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

n = 7

eg. In the binomial expansion of $(x+y)^7$,

the 3rd term of the expansion, $T_{2+1} = \binom{7}{2} x^{7-2} y^2$

$$= 21\, x^5 y^2$$

☑ Show full expansion of $(x+y)^7$,

$$= \binom{7}{0} x^7y^0 + \binom{7}{1} x^6y^1 + \underline{\binom{7}{2} x^5y^2} + \binom{7}{3} x^4y^3 + \binom{7}{4} x^3y^4 + \binom{7}{5} x^2y^5 + \binom{7}{6} x^1y^6 + \binom{7}{7} x^0y^7$$

$$= x^7 + 7x^6y + \underline{21x^5y^2} + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Illustrative Examples

Expand $(x + y)^4+(x - y)^4$ and hence find the value of $\left(\sqrt{2}+1\right)^4+\left(\sqrt{2}-1\right)^4$

Solution :

$$(x+y)^4 = {}^4C_0x^4y^0 + {}^4C_1x^3y^1 + {}^4C_2x^2y^2 + {}^4C_3x^1y^3 + {}^4C_4x^0y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Similarly $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$$

Hence $\left(\sqrt{2}+1\right)^4+\left(\sqrt{2}-1\right)^4 = 2\left(\sqrt{2}^4 + 6\sqrt{2}^2 1^2 + 1^4\right)$

$=34$

General Term in binomial expansion:

We have $(x + y)^n = nC_0\, x^n + nC_1\, x^{n-1} . y + nC_2\, x^{n-2} . y^2 + \ldots + nC_n\, y^n$

General Term = $T_{r+1} = nC_r\, x^{n-r} . y^r$

General Term in $(1 + x)^n$ is $nC_r\, x^r$

In the binomial expansion of $(x + y)^n$, the r^{th} term from end is $(n - r + 2)^{th}$.

Illustration: Find the number of terms in $(1 + 2x + x^2)^{50}$

Sol:

$(1 + 2x + x^2)^{50} = [(1 + x)^2]^{50} = (1 + x)^{100}$

The number of terms = (100 + 1) = 101

Illustration: Find the fourth term from the end in the expansion of $(2x - 1/x^2)^{10}$

Sol:

Required term $= T_{10-4+2} = T8 = 10C_7\, (2x)^3\, (-1/x^2)^7 = -960x^{-11}$

Middle Term(S) in the expansion of $(x+y)^n$

If n is even then $(n/2 + 1)$ Term is the middle Term.

If n is odd then $[(n+1)/2]^{th}$ and $[(n+3)/2)^{th}$ terms are the middle terms.

The Binomial Expansion

SUMMARY

- The binomial expansion of $(a+b)^n$ in ascending powers of x is given by

$$(a+b)^n =$$

$$^nC_0\, a^n \;+\; ^nC_1\, a^{n-1}b \;+\; ^nC_2\, a^{n-2}b^2 \;+\; . . . + b^n$$

- The ($r+1$) th term is $^nC_r\, a^{n-r}b^r$

- The expansion of $(1+x)^n$ is

$$(1+x)^n = {}^nC_0 \;+\; ^nC_1\, x \;+\; ^nC_2\, x^2 \;+\; . . . + x^n$$

Example 2

Find the coefficient of x^{-16} in the expansion of $(x^2 - \frac{1}{x})^{25}$

Solution

In the expansion, the (r + 1)th term

$$= {}^{25}C_r (x^2)^{25-r} (-\frac{1}{x})^r$$

$$= {}^{25}C_r x^{50-2r} (\frac{1}{x})^r (-1)^r$$

$$= {}^{25}C_r x^{50-2r} (x^{-r})(-1)^r$$

$$= {}^{25}C_r x^{50-3r} (-1)^r$$

To find coeff. of x^{-16}

$x^{-16} = x^{50-3r}$

$-16 = 50 - 3r$

$r = 22$

Coeff. of $x^{-16} = {}^{25}C_{22}(-1)^{22}$

$= 2300$

Sol. Given expansion is $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^{15}$

$$\therefore \qquad T_{r+1} = {}^{15}C_r \left(\frac{3x^2}{2}\right)^{15-r}\left(-\frac{1}{3x}\right)^r$$

$$\text{or} \qquad T_{r+1} = {}^{15}C_r\,(-1)^r\,3^{15-2r}2^{r-15}\,x^{30-3r} \qquad \text{(i)}$$

For the term independent of x, $30-3r=0 \quad \Rightarrow r=10$

$\therefore$ The term independent of x is

$$T_{10+1} = {}^{15}C_{10}\,3^{-5}\,2^{-5} \qquad \text{(Putting } r = 10 \text{ in (i))}$$

$$= {}^{15}C_{10}\left(\frac{1}{6}\right)^5$$

(ii) Given expression is $\left(3x - \frac{x^3}{6}\right)^9$

Here index, $n = 9$ (odd)

So, there are two middle terms, which are $\left(\frac{9+1}{2}\right)$th i.e., 5^{th} term and $\left(\frac{9+1}{2}+1\right)$th i.e., 6^{th} term.

$$\therefore \quad T_5 = T_{4+1} = {}^9C_4 (3x)^{9-4}\left(-\frac{x^3}{6}\right)^4$$

$$= \frac{9\times 8\times 7\times 6\times 5!}{4\times 3\times 2\times 1\times 5!} 3^5 x^5 x^{12} 6^{-4}$$

$$= \frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1}\times\frac{3^5}{3^4\times 2^4}x^{17} = \frac{189}{8}x^{17}$$

$$\text{And } T_6 = T_{5+1} = {}^9C_5 (3x)^{9-5}\left(-\frac{x^3}{6}\right)^5$$

$$= -\frac{9\times 8\times 7\times 6\times 5!}{5!\times 4\times 3\times 2\times 1}\cdot 3^4\cdot x^4\cdot x^{15}\cdot 6^{-5} = -\frac{-21}{16}x^{19}$$

Finding a Particular Term in a Binomial Expansion

a) Find the eighth term in the expansion of $(3x - 2)^{11}$.

$$t_{k+1} = {}^{n}C_k\, a^{n-k}\, b^k$$

$$t_{7+1} = {}^{11}C_7\, (3x)^{11-7}\, (-2)^7$$
$$t_8 = {}^{11}C_7\, (3x)^4\, (-2)^7$$
$$= 330(81x^4)(-128)$$
$$= -3\,421\,440\, x^4$$

$n = 11$
$a = 3x$
$b = -2$
$k = 7$

b) Find the middle term of $(a^2 - 3b^3)^8$.

$n = 8$, therefore, there are nine terms. The fifth term is the middle term.

$$t_{k+1} = {}^{n}C_k\, a^{n-k}\, b^k$$

$$t_{4+1} = {}^{8}C_4\, (a^2)^{8-4}\, (-3b^3)^4$$
$$t_5 = {}^{8}C_4\, (a^2)^4\, (-3b^3)^4$$
$$= 70a^8(81b^{12})$$
$$= 5670a^8b^{12}$$

$n = 8$
$a = a^2$
$b = -3b^3$
$k = 4$

$$\left(x^2 - \frac{1}{2x^7}\right)^{18}$$

Term independent of $x \Rightarrow x^0$

General Term:

$$\binom{18}{r}(x^2)^{18-r}\left(-\frac{1}{2x^7}\right)^r$$

$$= \binom{18}{r}\left(-\frac{1}{2}\right)^r (x^2)^{18-r}(x^{-7})^r$$

set power of $x = 0$

$$\therefore \quad 2(18-r) - 7r = 0$$

$$36 - 2r - 7r = 0$$

$$-9r = -36$$

$$r = 4$$

$\therefore$ Term independent of $x =$

$$\binom{18}{4}\left(-\frac{1}{2}\right)^4 = 191\tfrac{1}{4} \ \#$$

- Write the 5^{th} term from the end of the expansion $(x-1/x^2)^{12}$
- Sol:-The 5^{th} term from the end in the expansion of $(x-1/x^2)^{12}$ =The (12-5+2) i.e,9^{th} term from the beginning in the expansion of $(x-1/x^2)^{12}$
- Now $T_{r+1}={}^{n}Cr(x)^{n-r}(-1/x^2)^r$
- $T_{8+1}={}^{12}C8(x)^4(-1/x^2)^8$
- $=x^4(-1)^8(1/x^{16})$
- $=495\ x^{12}$

Fill in the blanks

1. The general term in the expansion of $(x+\frac{1}{x})^n$ ....
2. The number of term in the expansion of the binomial $(3x^2-\frac{1}{2y^2})^5$ is ......
3. If the number of term in the expansion of the binomial $(x-\frac{1}{3y^2})^n$ is 8 then the value of n is ....
4. n_{c_0} = ...
5. If $n_{c_4} = n_{c_5}$ then n = ...
6. If $8_{c_3} = 8_{c_r}$ then r = ...
7. Middle term in the expansion of $(2x+\frac{1}{2x})^4$ is ......
8. 10_{c_6} = ...
9. Sum of the coefficients in the expansion of $(2x^2-\frac{1}{3y^2})^5$ is ......
10. If $T_{r+1} = 8_{c_r} x^5 y^r$ then r = ...

Ans: 1. $T_{r+1} = n_{c_r} x^{n-r} (\frac{1}{x})^r$; 2. 6; 3. 7; 4. 1; 5. 9; 6. 5; 7. 4_{c_2}; 8. 210; 9. 32; 10. 3.

(Writer - V. Padma Priya)

a) the fifth member of the binomial expansion of the expression $(a-b)^6$

b) the third member of the binomial expansion of the expression $\left(\frac{1}{2}+\sqrt{2}\right)^5$

c) the fifth member of the binomial expansion of the expression $\left(x-\frac{1}{2}\sqrt{2}\right)^7$

d) the ninth member of the binomial expansion of the expression $\left(x^2-\frac{1}{x}\right)^{12}$

ALGEBRA

3. LOGARITHMS

Definition:-

Logarithmic Functions (continued)

The following definition demonstrates this connection between the exponential and the logarithmic function.

Definition of an Logarithmic Function

For $y > 0$, $b > 0$, and $b \neq 1$,

$$\text{If } y = b^x \text{, then } x = \log_b y$$

$y = b^x$ *is the* ***exponential form***

$x = \log_b y$ *is the* ***logarithmic form***

We read $\log_b y$ as "*log base b of y*".

Logarithmic Properties	
Product Rule	$\log_a(xy) = \log_a x + \log_a y$
Quotient Rule	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
Power Rule	$\log_a x^p = p\log_a x$
Change of Base Rule	$\log_a x = \frac{\log_b x}{\log_b a}$
Equality Rule	If $\log_a x = \log_a y$ then $x = y$

Exponential functions	Logarithmic functions	Read this way
$5^3 = 125$	$\log_5 125 = 3$	'log base 5 of 125 is 3'
$7^3 = 343$	$\log_7 343 = 3$	'log base 7 of 343 is 3'
$2^5 = 32$	$\log_2 32 = 5$	'log base 2 of 32 is 5'
$8^2 = 64$	$\log_8 64 = 2$	'log base 8 of 64 is 2'
$10^4 = 10{,}000$	$\log 10{,}000 = 4$	'log base 10 of 10,000 is 4'

EXAMPLE 6...

Solve the equation $\log_2 32 = 3x$ for x.

$$\log_2 32 = 3x$$

$$2^{3x} = 32$$ **Definition of Logarithms**

$$2^{3x} = 2^5$$ **Property of Equality**

$$3x = 5$$ **Equate the Exponents**

$$x = \frac{5}{3}$$ **Solve for x**

Simplify $\log_6 24 + 2 \log_6 3$

$= \log_6 24 + \log_6 3^2$ Power Property

$= \log_6 24 + \log_6 9$

$= \log_6 (24 \times 9)$ Product Property

$= \log_6 216$

$= 3$ $(6^3 = 216)$

$$
\begin{aligned}
\log_5 500 - 2\log_5 2 + \log_4 32 + \log_4 8 &= \log_5 500 - \log_5 2^2 + \log_4 32 + \log_4 8 \\
&= \log_5 500 - \log_5 4 + \log_4 32 + \log_4 8 \\
&= \log_5\left(\frac{500}{4}\right) + \log_4(32 \cdot 8) \\
&= \log_5(125) + \log_4(256) \\
&= \log_5\left(5^3\right) + \log_4\left(4^4\right) \\
&= 3 \cdot \log_5 5 + 4 \cdot \log_4(4) \\
&= 3(1) + 4(1) \\
&= 3 + 4 \\
&= 7
\end{aligned}
$$

$$\log_2 x + \log_4 x + \log_{16} x = 7$$

$$\log_2 x + \frac{\log_2 x}{\log_2 4} + \frac{\log_2 x}{\log_2 16} = 7$$

$$\log_2 x + \frac{\log_2 x}{2} + \frac{\log_2 x}{4} = 7 / .4$$

$$4\log_2 x + 2\log_2 x + \log_2 x = 28$$

$$7\log_2 x = 28$$

$$\log_2 x = 4$$

$$x = 2^4$$

$$x = 16$$

$$K = \{16\}$$

(*i*) $\log(2x+3) = \log 7$

$\Rightarrow \quad 2x + 3 = 7 \quad \Rightarrow \quad 2x = 7 - 3 \quad \Rightarrow \quad 2x = 4$

$\Rightarrow \quad x = \dfrac{4}{2} \quad \therefore \quad x = 2$

(*ii*) $\log(x+1) + \log(x-1) = \log 24$

$\Rightarrow \quad \log(x+1)(x-1) = \log 24$

$\Rightarrow \quad \log(x^2 - 1) = \log 24 \quad \Rightarrow \quad x^2 - 1 = 24$

$\Rightarrow \quad x^2 = 24 + 1 \quad \Rightarrow \quad x^2 = 25 \quad \Rightarrow \quad x^2 = (5)^2$

$\therefore \quad x^2 = 5$

(*iii*) $\log(10x+5) - \log(x-4) = 2$

$\Rightarrow \quad \log \dfrac{(10x+5)}{(x-4)} = 2(\log 10) \qquad [\therefore \ \log 10 = 1]$

$\Rightarrow \quad \log \dfrac{10x+5}{x-4} = \log(10)^2$

$\Rightarrow \quad \log\left(\dfrac{10x+5}{x-4}\right) = \log 100 \quad \Rightarrow \quad \dfrac{10x+5}{(x-4)} = 100$

$\Rightarrow \quad 10x + 5 = 100(x-4)$

$\Rightarrow \quad 10x + 5 = 100x - 400 \Rightarrow 10x - 100x = -400 - 5$

$\Rightarrow \quad -90x = -405 \quad \Rightarrow \quad x = \dfrac{-405}{-90}$

$\Rightarrow \quad x = \dfrac{405}{90} = \dfrac{81}{18} = \dfrac{9}{2} \quad \therefore \quad x = 4.5$

$$\log_x 25 - \log_x 5 = 2 - \log_x \frac{1}{125}$$

$$\Rightarrow \log_x 5^2 - \log_x 5 = 2 - \log_x \left(\frac{1}{5}\right)^3$$

$$\Rightarrow \log_x 5^2 - \log_x 5 = 2 - \log_x 5^{-3}$$

$$\Rightarrow 2\log_x 5 - \log_x 5 = 2 + 3\log_x 5$$

$$\Rightarrow 2\log_x 5 - \log_x 5 - 3\log_x 5 = 2$$

$$\Rightarrow -2\log_x 5 = 2$$

$$\Rightarrow \log_x 5 = -1$$

$$\Rightarrow x^{-1} = 5$$

$$\Rightarrow \frac{1}{x} = 5$$

$$\Rightarrow x = \frac{1}{5}$$

Exponential Equations: Strategies

Anil Kumar

Solve: $2^{x^2} = 32\left(2^{4x}\right)$

$2^{x^2} = 2^5\left(2^{4x}\right)$

$2^{x^2} = 2^{5+4x}$

$x^2 = 5 + 4x$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$32 = 2^5$

$(x-5)(x+1) = 0$

$x = 5$ | $x = -1$

check.

$2^{5^2} = 2^{5+4(5)}$

$2^{25} = 2^{25}$

✓

EXAMPLE 9 Using the Product and Quotient Rules

Solution continued

b. $$\begin{aligned} \log_2(x+4) + \log_2(x+3) &= 1 \\ \log_2[(x+4)(x+3)] &= 1 \\ (x+4)(x+3) &= 2^1 \\ x^2 + 7x + 10 &= 0 \\ (x+2)(x+5) &= 0 \\ x = -2 \quad \text{or} \quad x &= -5 \end{aligned}$$

EXAMPLE 7 Solving a Logarithmic Equation

Solve: $4 + 3\log_2 x = 1$.

Solution

$$4 + 3\log_2 x = 1$$
$$3\log_2 x = 1 - 4 = -3$$
$$\log_2 x = -1$$
$$x = 2^{-1}$$
$$x = \frac{1}{2}$$

We must check our solution.

$$Solve\, 2\log 5 + \log(x+3) = 2$$

$$\log 5^2 + \log(x+3) = 2$$

$$\log 5^2 (x+3) = 2$$

$$\log_{10} 25(x+3) = 2$$

$$10^2 = 25(x+3)$$

$$100 = 25x + 75$$

$$25 = 25x$$

$$1 = x$$

Solving a Logarithmic Equation

- Example: Solve $\log(x+6)-\log(x+2)=\log x$

$$\log\frac{x+6}{x+2}=\log x$$

$$\frac{x+6}{x+2}=x$$

$$x+6=x(x+2)$$

$$x+6=x^2+2x$$

$$0=x^2+x-6$$

$$x^2+x-6=0$$

$$(x+3)(x-2)=0$$

$$x=-3,\ 2$$

If we plug in –3, x+2 is negative. We can't take the log of a negative number, so our answer is 2.

ALGEBRA

4. MATRCIES AND DETERMINANTS

DEFINITION OF MATRCIES

Matrix Algebra

- **Definition**: ***A matrix is a rectangular or square array of elements (usually numbers) arranged in rows and columns.***
- Matrices are usually shown by capital and bold letters such as ***A***, ***B***, etc. Matrix ***A*** with 3 rows and 2 columns is shown by $\boldsymbol{A_{3\times 2}}$ and matrix ***B*** with m rows and n columns is shown by $\boldsymbol{B_{m\times n}}$. Their elements are shown by small letters with an index indicating the position of the element in the matrix.
- $A_{3\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \quad B_{m\times n} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} \dots & b_{2n} \\ \vdots & \vdots & \vdots \quad \dots & \vdots \\ b_{m1} & b_{m2} & b_{m3} \cdots & b_{mn} \end{pmatrix}$

No. of Columns 'n'

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

No. of Rows 'm'

Types of Matrices

Row Matrix

$$\begin{pmatrix} a & b & c \end{pmatrix}$$

Column Matrix

Vector Matrix

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Zero Matrix

Null Matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Diagonal Matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Scalar Matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

Unit Matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Upper Triangular Matrix

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

Lower Triangular Matrix

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

Table 1: Matrix Properties

Properties of Matrices	
ADDITION	
$A + B = B + A$	Commutativity
$A + (B + C) = (A + B) + C$	Associativity
$A + Z = A$	Zero matrix exists
MULTIPLICATION	
$AB \neq BA$	Non-Commutativity
$(AB)C = A(BC)$	Associativity
$AI = A$	Identity matrix exists
$A(B + C) = AB + AC$	Distributivity over addition
$(B + C)D = BD + CD$	
$A(kB) = kAB$	Associativity over scalar multiplication
INVERSE AND IDENTITY	
$AI = IA = A$	
$A^{-1}A = AA^{-1} = I$	

In this table, A, B, and C are $n \times n$ matrices, I is the $n \times n$ identity matrix, and O is the $n \times n$ zero matrix

Property	Example
The commutative property of multiplication does not hold!	$AB \neq BA$
Associative property of multiplication	$(AB)C = A(BC)$
Distributive properties	$A(B+C) = AB + AC$
	$(B+C)A = BA + CA$
Multiplicative identity property	$IA = A$ and $AI = A$
Multiplicative property of zero	$OA = O$ and $AO = O$
Dimension property	The product of an $m \times n$ matrix and an $n \times k$ matrix is an $m \times k$ matrix.

Transpose Matrices

DEFINITION 6:

Let A = $[a_{ij}]$ be an m x n matrix. The **transpose** of A, denoted by $\mathbf{A}^t$, is the n x m matrix obtained by interchanging the rows and columns of A.

In other words, if

$$A^t = [b_{ij}], \text{ then } b_{ij} = a_{ji}, \text{ for } i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m.$$

- **Example:**

  The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- **Example 2:**
- Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$, Find A^t.

$$A^T = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & -2 \end{bmatrix}$$

Matrices - Operations

Properties of transposed matrices:

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
3. $(k\mathbf{A})^T = k\mathbf{A}^T$
4. $(\mathbf{A}^T)^T = \mathbf{A}$

Symmetric and Skew – Symmetric Matrix

A square matrix A is called a symmetric matrix, if $A^T = A$.

A square matrix A is called a skew- symmetric matrix, if $A^T =$ A.

Any square matrix can be expressed as the sum of a symmetric and a skew- symmetric matrix.

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2},$$

where $\left(A + A^T\right)$ is symmetric matrix and $\left(A - A^T\right)$ is skew - symmetric matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+9 & 2+8 & 3+7 \\ 4+6 & 5+5 & 6+4 \\ 7+3 & 8+2 & 9+1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}$$

Two matrices with the same dimensions can be added or subtracted, by finding the sums or differences of the corresponding elements.

EX: •

$$A = \begin{bmatrix} 3 & 8 & 1 \\ 4 & 0 & -3 \\ -2 & 1 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 9 \\ 4 & -6 & -5 \\ 0 & 7 & 2 \end{bmatrix}$$

Find A + B and A – B.

$$A + B = \begin{bmatrix} 5 & 8 & 10 \\ 8 & -6 & -8 \\ -2 & 8 & 7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 8 & -8 \\ 0 & 6 & 2 \\ -2 & -6 & 3 \end{bmatrix}$$

Example 20

Find minors and cofactors of all the elements of the determinant
$\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Minor of a_{11} = M_{11} = $\begin{vmatrix} \cancel{1} & \cancel{-2} \\ \cancel{4} & 3 \end{vmatrix}$ = 3

Minor of a_{12} = M_{12} = $\begin{vmatrix} \cancel{1} & \cancel{2} \\ 4 & \cancel{3} \end{vmatrix}$ = 4

Minor of a_{21} = M_{21} = $\begin{vmatrix} \cancel{1} & -2 \\ \cancel{4} & \cancel{3} \end{vmatrix}$ = −2

Minor of a_{22} = M_{22} = $\begin{vmatrix} 1 & \cancel{-2} \\ \cancel{4} & \cancel{3} \end{vmatrix}$ = 1

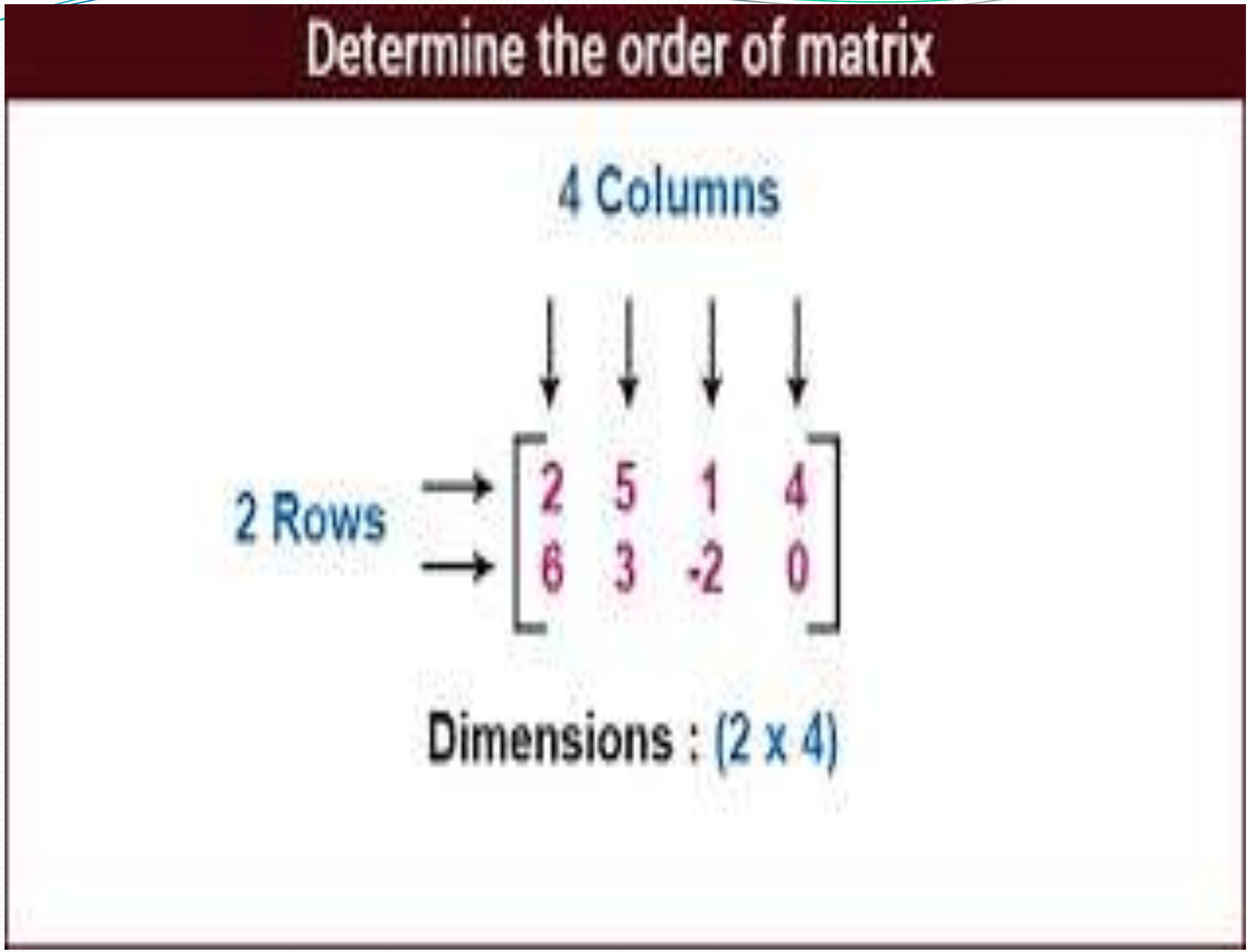
$$\boxed{\textit{Cofactor of } a_{ij} = A_{ij} = (-1)^{i+j}.\ M_{ij}}$$

Cofactor of a_{11} = A_{11} = $(-1)^{1+1}$ M_{11} = $(-1)^2 . 3$ = 3

Cofactor of a_{12} = A_{12} = $(-1)^{1+2}$ M_{12} = $(-1)^3$. 4 = (−1) (4) = −4

Cofactor of a_{21} = A_{21} = $(-1)^{2+1}$. M_{21} = $(-1)^3$ (−2) = (−1) (−2) = 2

Cofactor of a_{22} = A_{22} = $(-1)^{2+2}$. M_{22} = $(-1)^4$. (1) = (1) (1) = 1


Determine the order of matrix
4 Columns
2 Rows
2 5 1 4
6 3 -2 0
Dimensions : (2 x 4)

How to express each square matrix as sum of symmetric and skew symmetric matrix.

$$A=\begin{bmatrix}3 & 7\\ 9 & -5\end{bmatrix} \Rightarrow A^{T}=\begin{bmatrix}3 & 9\\ 7 & -5\end{bmatrix}$$

$$Let\ P=\frac{1}{2}(A+A^{T})=\frac{1}{2}\left(\begin{bmatrix}3 & 7\\ 9 & -5\end{bmatrix}+\begin{bmatrix}3 & 9\\ 7 & -5\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}6 & 16\\ 16 & -10\end{bmatrix}=\begin{bmatrix}3 & 8\\ 8 & -5\end{bmatrix}$$

$$Now,\ P^{T}=\begin{bmatrix}3 & 8\\ 8 & -5\end{bmatrix}=P \Rightarrow P\ is\ a\ symmetric\ matrix.$$

$$and\ let\ Q=\frac{1}{2}(A-A^{T})=\frac{1}{2}\left(\begin{bmatrix}3 & 7\\ 9 & -5\end{bmatrix}-\begin{bmatrix}3 & 9\\ 7 & -5\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$and,\ Q^{T}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=-\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=-Q \Rightarrow Q\ is\ a\ skew\ symmetric\ matrix.$$

$$P+Q=\begin{bmatrix}3 & 8\\ 8 & -5\end{bmatrix}+\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}3 & 7\\ 9 & -5\end{bmatrix}=A$$

Matrix Addition and Subtraction: Two matrices must be exactly the same size; otherwise addition or subtraction is **undefined**. When adding or subtracting matrices, add (or subtract) the corresponding elements of the matrices. (Subtraction is just adding the opposite matrix)

Addition Examples

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} (a+w) & (b+x) \\ (c+y) & (d+z) \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -3 & 2 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (-2-1) & (3+2) \\ (0-3) & (1+2) \\ (2+5) & (-1-2) \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -3 & 3 \\ 7 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

The sum is undefined.
First matrix is {3x2}
Second is {2x2}

Subtraction Examples

$$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 5 & -3 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (3-5) & (-1-(-3)) \\ (2-4) & (0-(-1)) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -3 & 2 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (-2+1) & (3-2) \\ (0+3) & (1-2) \\ (2-5) & (-1+2) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -1 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

The difference is undefined.
First matrix is {3x2}
Second is {2x2}

$$4.\ If\ A=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{bmatrix} and\ B=\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix},$$

$$find\ \ 3B-2A.$$

Solution :

$$A=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{bmatrix}, B=\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}$$

$$2B-2A=3\begin{bmatrix}3 & 2 & 1\\ 1 & 2 & 3\end{bmatrix}-2\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}9 & 6 & 3\\ 3 & 6 & 9\end{bmatrix}-\begin{bmatrix}2 & 4 & 6\\ 6 & 4 & 2\end{bmatrix}$$

$$=\begin{bmatrix}9-2 & 6-4 & 3-6\\ 3-6 & 6-4 & 9-2\end{bmatrix}=\begin{bmatrix}7 & 2 & -3\\ -3 & 2 & 7\end{bmatrix}$$

Equality of Matrices

$$\begin{bmatrix} 3 & x+y \\ x-y & 5 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$$

Find x and y.

Multiplication of two matrcies:- if columns of Ist Matrix=Rows of IInd matrix

$$c_{11}=a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41}$$

$$\underset{2 \times 4}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}} \underset{4 \times 3}{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}} = \underset{2 \times 3}{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}}$$

$$c_{22}=a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + (-2) \times 3 + 1 \times 1 & 1 \times 1 + (-2) \times 2 + 1 \times 1 \\ 2 \times 2 + 1 \times 3 + 3 \times 1 & 2 \times 1 + 1 \times 2 + 3 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -2 \\ 10 & 7 \end{bmatrix}$$

For $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}$

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 5 & 9 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 4 \\ -3 & 1 \end{bmatrix}$$

$$A(B+C) = AB + AC$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\left(\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix} + \begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix} + \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 3\\1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & 3\\2 & 3\end{bmatrix} = \begin{bmatrix}3 & 4\\7 & 8\end{bmatrix} + \begin{bmatrix}3 & 5\\7 & 13\end{bmatrix}$$

$$\begin{bmatrix}6 & 9\\14 & 21\end{bmatrix} = \begin{bmatrix}6 & 9\\14 & 21\end{bmatrix}$$

Given: $A = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 2 \\ -1 & 4 \end{bmatrix}$

Now,

$$A^2 = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4+0 & 2-2 \\ 0+0 & 0+4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -6-1 & 4+4 \\ 0+2 & 0-8 \end{bmatrix} = \begin{bmatrix} -7 & 8 \\ 2 & -8 \end{bmatrix}$$

$$5B = 5\begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 20 & 5 \\ -15 & -10 \end{bmatrix}$$

$$\therefore\ A^2 + AC - 5B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -7 & 8 \\ 2 & -8 \end{bmatrix} - \begin{bmatrix} 20 & 5 \\ -15 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 4-7-20 & 0+8-5 \\ 0+2+15 & 4-8+10 \end{bmatrix}$$

$$= \begin{bmatrix} -23 & 3 \\ 17 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$$

Q1(i) If a matrix has 12 elements, what are the possible orders it can have ? What if it has 7 elements?

(ii) If a matrix has 8 elements, what are the possible orders it can have? What if it has 5 elements?

Q2. Construct a 2x3 matrix whose elements in the eth row and jth column is given by :-

(i) $aij = \frac{i+3j}{2}$ (i) $aij = \frac{2i+3j}{2}$ (iii) $aij = \frac{3i+j}{2}$ (iv) $aij = \frac{3i-j}{2}$

Q3. Construct a 4x3 matrix whose elements are:-

(i) $aej = 2i + \frac{e'}{f}$ (ii) $aej = \frac{i-j}{j+j}$ (iii) $aej = i$

Q4. If

$$\begin{pmatrix} 2+3 & 2+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z=2c \end{pmatrix} = \begin{pmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c-2 \\ 2b+4 & -21 & 0 \end{pmatrix}$$

Obtain the values of a, b, c, x, y and z.

Q5. Find matrices x and y i.e.

$$2x-y = \begin{pmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{pmatrix} \quad \text{and } x + 2y = \begin{pmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{pmatrix}$$

Q6. Find the value of x such that :-

$$[1 \quad 1 \quad x] \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

Q7. If $A = \begin{pmatrix} 3 & -4, \\ 1 & -1 \end{pmatrix}$ prove that $A^n = \begin{pmatrix} 1+2n & -4n, \\ n & 1-2n \end{pmatrix}$ where n is any positive integer.

Q8. If $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -4 & 5 \\ 0 & -1 & 3 \end{pmatrix}$, find $A^2 - 4A + 3I3$

Q9. Express the matrix $A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{pmatrix}$ as the sum of a symmetric and a skew symmetric matrix

Q10. Express the following matrices as the sum of symmetric and skew-symmetric matrices:-

(i) $A = \begin{pmatrix} 6 & 1 \\ 3 & 4 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{pmatrix}$

(iii) $A = \begin{pmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 6 & 0 & 8 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{pmatrix}$

3.1 The Determinant of a Matrix

- Every *square* matrix can be associated with a *real* number called its **determinant**.
- ***Definition***: The determinant of the matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

  is given by $\det(A)=|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$

  $-$ $+$
- ***Example 1***: $\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}=2(2)-1(-3)=7$ $\quad A=[-2]\Rightarrow |A|=-2$

  $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}=2(2)-1(4)=0$

  $\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix}=0(4)-2(3)=-6$

Example 1.6 :
We can compute the determinant $|\mathbf{T}|=\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

by expanding along the first row,

$$|\mathbf{T}|=1\times(-)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}+2\times(-)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}+3\times(-)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \quad =-3+12-9=0$$

Or expand down the second column:

$$|\mathbf{T}|=2\times(-)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}+5\times(-)^{2+2}\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}+8\times(-)^{3+2}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \quad =12-60+48=0$$

Example 1.7: A row or column with many zeroes suggests a Laplace expansion.

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix}=1\times(-)^{2+3}\begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} \quad =16$$

1) $\begin{vmatrix} -2 & 3 & 1 \\ 4 & -2 & 0 \\ 1 & -2 & 3 \end{vmatrix}$

Sign array. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$$= +1\begin{vmatrix} 4 & -2 \\ 1 & -2 \end{vmatrix} - 0\cancel{\begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix}} + 3\begin{vmatrix} -2 & 3 \\ 4 & -2 \end{vmatrix}$$

$$= 1\left(4(-2) - (-2)(1)\right) + 3\left((-2)(-2) - (3)(4)\right)$$

$$= 1\left(-8 - (-2)\right) + 3\left(4 - 12\right)$$

$$= -8 + ^{\frown}$$

Cramer's Rule for Three Equations in Three Unknowns

The solution to the system

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

is given by $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, and $z = \frac{D_z}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

provided that $D \neq 0$.

$$\left.\begin{aligned} 2x + y &= -1 \\ 4x + 3y &= 1 \end{aligned}\right\}$$

$$D = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 2 \cdot 3 - 4 \cdot 1 = 6 - 4 = 2,$$

$$x = \frac{D_x}{D} = \frac{-4}{2} = -2,$$

$$D_x = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = -1 \cdot 3 - 1 \cdot 1 = -3 - 1 = -4,$$

$$y = \frac{D_y}{D} = \frac{6}{2} = 3.$$

$$D_y = \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = 2 \cdot 1 - 4 \cdot (-1) = 2 + 4 = 6,$$

$$\begin{cases} 3x+2y-z=1 \\ x-y+5z=-2 \\ 2x+y=3 \end{cases} \quad \text{is a Cramer's system}$$

$$x=\frac{\begin{vmatrix} 1 & 2 & -1 \\ -2 & -1 & 5 \\ 3 & 1 & 0 \end{vmatrix}}{\det A}=\frac{24}{2}=12; \quad y=\frac{\begin{vmatrix} 3 & 1 & -1 \\ 1 & -2 & 5 \\ 2 & 3 & 0 \end{vmatrix}}{\det A}=\frac{-42}{2}=-21$$

$$z=\frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & -2 \\ 2 & 1 & 3 \end{vmatrix}}{\det A}=\frac{-14}{2}=-7 \qquad \begin{cases} x=12 \\ y=-21 \\ z=-7 \end{cases}$$

We expand by minors about row 1 to find D.

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

Given system:

$$\begin{aligned} x + y - z &= -2 \\ 2x - y + z &= -5 \\ x - 2y + 3z &= 4 \end{aligned}$$

$$= 1\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= 1(-1) - 1(5) - 1(-3)$$

$$= -3$$

Expand D_x by minors about row 1.

$$D_x = \begin{vmatrix} -2 & 1 & -1 \\ -5 & -1 & 1 \\ 4 & -2 & 3 \end{vmatrix}$$

$$= -2\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} - 1\begin{vmatrix} -5 & 1 \\ 4 & 3 \end{vmatrix} + (-1)\begin{vmatrix} -5 & -1 \\ 4 & -2 \end{vmatrix}$$

$$= -2(-1) - 1(-19) - 1(14)$$

$$= 7$$

Verify that $D_y = -22$ and $D_z = -21$. Thus,

$$x = \frac{D_x}{D} = \frac{7}{-3} = -\frac{7}{3}, \qquad y = \frac{D_y}{D} = \frac{-22}{-3} = \frac{22}{3}, \qquad z = \frac{D_z}{D} = \frac{-21}{-3} = 7.$$

Check that the solution set is $\left\{\left(-\frac{7}{3}, \frac{22}{3}, 7\right)\right\}$.

Name : ______________________ Score : ______________

Teacher : ______________________ Date : ______________

Cramers Rule with System of 2 Equations

Use Cramers Rule to solve each system.

1) $-2x - 2y = 19$
$4x + 4y = 4$

2) $-5x - 2y = -7$
$3x - 3y = -21$

3) $4x + 4y = 16$
$3x + 3y = 23$

4) $4x + 3y = -26$
$3x + 4y = -30$

5) $-2x + 6y = 4$
$6x + 5y = -35$

6) $-3x - 2y = -19$
$3x + 2y = 19$

7) $-3x - 5y = -1$
$4x + 2y = 20$

8) $6x + 6y = -46$
$x + y = 17$

9) $-x - 2y = -9$
$-2x - 3y = -11$

10) $-6x - 3y = -15$
$-x - 5y = -16$

11) $-5x + 6y = 60$
$2x - 6y = -42$

12) $-4x - 4y = -20$
$-2x - 2y = -10$

Solve the following systems of equations by Cramer's rule.

(a) $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2, \end{cases}$

where $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

(b) $\begin{cases} 2x_1 + x_2 - 3x_3 = 5 \\ x_1 - 2x_2 + x_3 = 10 \\ 3x_1 + 4x_2 - 2x_3 = 0 \end{cases}$

(c) $\begin{cases} 2x_1 + x_2 - 3x_3 = 1 \\ x_1 - 2x_2 + x_3 = 0 \\ 3x_1 + 4x_2 - 2x_3 = -5 \end{cases}$

(d) $\begin{cases} x_1 - x_2 + 4x_3 = -4 \\ -8x_1 + 3x_2 + x_3 = 8 \\ 2x_1 - x_2 + x_3 = 0 \end{cases}$

(e) $\begin{cases} x_1 - x_2 + 4x_3 = -2 \\ -8x_1 + 3x_2 + x_3 = 0 \\ 2x_1 - x_2 + x_3 = 6 \end{cases}$

(f) $\begin{cases} 3x_1 + x_2 + x_3 = 4 \\ -2x_1 - x_2 = 12 \\ x_1 + 2x_2 + x_3 = -8 \end{cases}$

ALGEBRA

5. COMPLEX NUMBERS

Definition of iota

Iota is a greek letter which is widely used in mathematics to denote the imaginary part of a complex number. Let's say we have an equation: $x^2 + 1 = 0$. In this case, the value of x will be the square root of -1, which is fundamentally not

Integral powers of i(iota)

$i^0 = 1$ (as usual)

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 . i = -i$

$i^4 = i^3 . i = -i . i = 1$

$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$

$i^{-2} = \frac{1}{i^2} = -1$

$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} = i$

$i^{-4} = \frac{1}{i^4} = 1$

Evaluate:

$$\left(i^{17} - \left(\frac{2}{i}\right)^3\right)^3$$

Solution

$$\left(i^{16} . i - \frac{8}{i^3}\right)^3 = \left(i + \frac{8}{i}\right)^3 = (i - 8i)^3$$

Ans: 343i

CAREER LAUNCHER

$$i^{1000}$$
$$=(i^2)^{500}$$
$$=(-1)^{500}$$
$$=\left((-1)^2\right)^{250}$$
$$=(1)^{250}=1$$

10 minutes – have a go!

Complex Numbers - Introduction

Express in terms of i:

1. $\sqrt{(-64)}$ $= \pm 8i$

2. $\sqrt{(-7)}$ $= \pm\sqrt{7}\ i$

3. $\sqrt{16} - \sqrt{(-81)}$ $= \pm 4 \mp 9i$

4. $3 - \sqrt{(-25)}$ $3 \pm 5i$

5. $\sqrt{(-100)} - \sqrt{(-49)}$ $\pm 10i \pm 7i$

Simplify

1. i^3 $= -i$

2. i^7 $= -i$

3. i^{-9} $= \frac{1}{i^9} = \frac{1}{(-1)(-1)(-1)(-1)i} = \frac{1}{i} \times \frac{i}{i} = -i$

4. $i(2i - 3i^3)$ $= 2i^2 - 3i^4$
$= -2 - 3$
$= \underline{\underline{-5}}$

5. $(i + 2i^2)(3 - i)$ $= 3i - i^2 + 6i^2 - 2i^3$
$= 3i + 1 - 6 + 2i$
$= -5 + 5i$

Definition

- A **complex number** z is a number of the form

$$x+jy \quad \text{where} \quad j=\sqrt{-1}$$

- x is the real part and y the imaginary part, written as $x =$ **Re** z, $y =$ **Im** z.

- j is called the imaginary unit

- If $x = 0$, then $z = jy$ is a pure imaginary number.

- The **complex conjugate** of a complex number, $z = x + jy$, denoted by z^* , is given by

$$z^* = x - jy.$$

- Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

The complex numbers satisfy the following properties under **addition.**	The complex numbers satisfy the following properties under **multiplication.**
(i) **Closure property** For any two complex numbers z_1 and z_2, the sum z_1+z_2 is also a complex number.	(i) **Closure property** For any two complex numbers z_1 and z_2, the product $z_1 z_2$ is also a complex number.
(ii) **The commutative property** For any two complex numbers z_1 and z_2 $z_1+z_2=z_2+z_1$.	(ii) **The commutative property** For any two complex numbers z_1 and z_2 $z_1z_2=z_2z_1$.
(iii) **The associative property** For any three complex numbers z_1, z_2, and z_3 $(z_1+z_2)+z_3=z_1+(z_2+z_3)$.	(iii) **The associative property** For any three complex numbers z_1, z_2, and z_3 $(z_1z_2)z_3=z_1(z_2z_3)$.
(iv) **The additive identity** There exists a complex number $0=0+0i$ such that, for every complex number z, $z+0=0+z=z$ The complex number $0=0+0i$ is known as additive identity.	(iv) **The multiplicative identity** There exists a complex number $1=1+0i$ such that, for every complex number z, $z1=1z=z$ The complex number $1=1+0i$ is known as multiplicative identity.
(v) **The additive inverse** For every complex number z there exists a complex number $-z$ such that, $z+(-z)=(-z)+z=0.$ $-z$ is called the additive inverse of z.	(v) **The multiplicative inverse** For any nonzero complex number z, there exists a complex number w such that, $zw=wz=1.$ w is called the multiplicative inverse of z. w is denoted by z^{-1}.
(vi) **Distributive property (multiplication distributes over addition)** For any three complex numbers z_1, z_2, and z_3 $z_1(z_2+z_3)=z_1z_2+z_1z_3$ and $(z_1+z_2)z_3=z_1z_3+z_2z_3$.	

ABSOLUTE VALUE

The *absolute value* or *modulus* of a complex number $a+bi$ is defined as $|a+bi| = \sqrt{a^2+b^2}$.

Example: $|-4+2i| = \sqrt{(-4)^2+(2)^2} = \sqrt{20} = 2\sqrt{5}$

If $z_1, z_2, z_3, \ldots, z_m$ are complex numbers, the following properties hold.

1. $|z_1 z_2| = |z_1||z_2|$ or $|z_1 z_2 \cdots z_m| = |z_1||z_2| \cdots |z_m|$
2. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \neq 0$
3. $|z_1+z_2| \leqq |z_1|+|z_2|$ or $|z_1+z_2+\cdots+z_m| \leqq |z_1|+|z_2|+\cdots+|z_m|$
4. $|z_1+z_2| \geqq |z_1|-|z_2|$ or $|z_1-z_2| \geqq |z_1|-|z_2|$

Complex Conjugates

- If $z = a+bi$ is any complex number, then the *complex conjugate* of z (also called the conjugate of z) is denoted by the symbol $\overline{Z}$ and is defined by :

$$\overline{Z} = a - bi$$

Properties of Conjugate:

- $\overline{(\bar{z})} = z$
- $|z| = |\bar{z}|$
- $z + \bar{z} = 2Re(z)$. $z - \bar{z} = 2i\ Im(z)$.
- If z is purely real $z = \bar{z}$. whenever we have to show a complex number purely real we use this property.
- If z is purely imaginary $z + \bar{z} = 0$, whenever we have to show that a complex number is purely imaginary we use this property.
- $z\bar{z} = |z|^2 = |\bar{z}|^2$
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

  In general, $\overline{z_1 + z_2 + ... + z_n} = \bar{z}_1 + \bar{z}_2 + ... + \bar{z}_n$
- $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

  In general $\overline{z_1 z_2 z_3 .....z_n} = \bar{z}_1 . \bar{z}_2 . \bar{z}_3 .....\bar{z}_n$
- $\overline{z^n} = (\bar{z})^n$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{\bar{z}_1}{\bar{z}_2}\right)$

$$
\begin{aligned}
(3+2i)(1-4i) &= 3-12i+2i-8i^2 \\
&= 3-10i-8\left(\sqrt{-1}\right)^2 \\
&= 3-10i-8(-1) \\
&= 3-10i+8 \\
&= 11-10i
\end{aligned}
$$

$$\frac{2+3i}{4-5i}=\frac{2+3i}{4-5i}\cdot\frac{4+5i}{4+5i}=$$

$$=\frac{8+10i+12i+15i^2}{16+20i-20i-25i^2}=$$

$$=\frac{8+22i+15\cdot(-1)}{16-25\cdot(-1)}=$$

$$=\frac{-7+22i}{49}=$$

$$=-\frac{7}{49}+\frac{22}{49}i$$

$$\left(\frac{-2}{1+i}\right)\left(\frac{1-i}{1-i}\right)=\frac{-2+2i}{1-i+i-i^2}$$

$$=\frac{-2+2i}{1-i^2}$$

$$=\frac{-2+2i}{1-(-1)}$$

$$=\frac{-2+2i}{1+1}$$

$$=\frac{-2+2i}{2}$$

$$=\frac{2(-1+i)}{2}$$

$$\left(\frac{-2}{1+i}\right)\left(\frac{1-i}{1-i}\right)=-1+i$$

- **Simplify $(2 + 3i) + (1 - 6i)$.**

$(2 + 3i) + (1 - 6i) = (2 + 1) + (3i - 6i) = 3 + (-3i) = \mathbf{3 - 3i}$

- **Simplify $(5 - 2i) - (-4 - i)$.**

$(5 - 2i) - (-4 - i)$

$= (5 - (-4)) + (-2i - (-i)) = (5 + 4) + (-2i + i)$

$= (9) + (-1i) = \mathbf{9 - i}$

- **Simplify $(2 - i)(3 + 4i)$.**

$(2 - i)(3 + 4i) = (2)(3) + (2)(4i) + (-i)(3) + (-i)(4i)$

$= 6 + 8i - 3i - 4i^2 = 6 + 5i - 4(-1)$

$= 6 + 5i + 4 = \mathbf{10 + 5i}$

4. If $\dfrac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

Sol. $$x + iy = \frac{(1+i)^2}{2-i} = \frac{1+2i+i^2}{2-i} = \frac{2i}{2-i} = \frac{2i(2+i)}{(2-i)(2+i)} = \frac{4i+2i^2}{4-i^2}$$

$$= \frac{4i-2}{4+1} = \frac{-2}{5} + \frac{4i}{5}$$

$$\Rightarrow \qquad x = \frac{-2}{5}, y = \frac{4}{5} \Rightarrow x + y = \frac{-2}{5} + \frac{4}{5} = \frac{2}{5}$$

5. If $\left(\dfrac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

Sol. $$a + ib = \left(\frac{1-i}{1+i}\right)^{10} = \left[\frac{(1-i)}{(1+i)} \cdot \frac{(1-i)}{(1-i)}\right]^{100} = \left[\frac{(1-i)^2}{1-i^2}\right]^{100}$$

$$= \left(\frac{1-2i+i^2}{1+1}\right)^{100} = \left(\frac{-2i}{2}\right)^{100} = (i^4)^{25} = 1$$

$$\therefore \qquad (a, b) = (1, 0)$$

Find the complex conjugate of $-7-i$ and the sum of this number with its complex conjugate.

$z = a + bi$ $\qquad$ $z^* = a - bi$

$z = -7 - i$ $\qquad$ $z^* = -7 + i$

$z + z^* = (-7 - i) + (-7 + i)$

$= -14$

$\boxed{-7+i, -14}$

$z + z^* = 2\,\mathrm{Re}(z)$

$a + \cancel{bi} + a - \cancel{bi} = 2a$

Ex 3: Determine the complex conjugates of $z = 4 + 3i$ and $w = 2 - 5i$ and then find:

$$\overline{z} + \overline{w},\ \overline{z + w},\ \overline{z} \cdot \overline{w},\ \overline{z \cdot w}$$

Solution: $\overline{z} = 4 - 3i$ $\quad\quad$ $\overline{w} = 2 + 5i$

$$\overline{z} + \overline{w} = (4 - 3i) + (2 + 5i) = 6 + 2i$$

$$\overline{z + w} = \overline{(4 + 3i) + (2 - 5i)} = \overline{6 - 2i} = 6 + 2i$$

$$\overline{z} \cdot \overline{w} = (4 - 3i)(2 + 5i) = 8 + 20i - 6i - 15i^2 = 23 + 14i$$

$$\overline{z \cdot w} = \overline{(4 + 3i)(2 - 5i)} = \overline{8 - 20i + 6i - 15i^2} = \overline{23 - 14i} = 23 + 14i$$

Conjugate Results

For any pair of complex numbers, z and w, and for any integer n, we have

$$\overline{z} + \overline{w} = \overline{z + w}, \quad \overline{z} \cdot \overline{w} = \overline{z \cdot w} \ \ and \ \ \overline{z}^n = \overline{z^n}$$

iTutor

Absolute Value of a Complex Number

- The distance the complex number is from the origin on the complex plane.
- If you have a complex number $(a+bi)$

the absolute value can be found using:

$$\sqrt{a^2+b^2}$$

Examples

1. $|-2+5i|$

$$=\sqrt{(-2)^2+(5)^2}$$

$$=\sqrt{4+25}$$

$$=\sqrt{29}$$

2. $|-6i|$

$$=\sqrt{(0)^2+(-6)^2}$$

$$=\sqrt{0+36}$$

$$=\sqrt{36}$$

$$=6$$

Example 2: Determining the Absolute Value of Complex Numbers

Find each absolute value.

A. $|3 + 5i|$

$\sqrt{3^2 + 5^2}$

$\sqrt{9 + 25}$

$\sqrt{34}$

B. $|-13|$

$|-13 + 0i|$

$\sqrt{(-13)^2 + 0^2}$

$\sqrt{169}$

13

C. $|-7i|$

$|0 + (-7)i|$

$\sqrt{0^2 + (-7)^2}$

$\sqrt{49}$

7

Let the given complex number be $z = \dfrac{1+2i}{1-3i}$

$$z = \frac{1+2i}{1-3i}$$

$$\Rightarrow z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$\Rightarrow z = \frac{1+3i+2i+6i^2}{(1)^2-(3i)^2}$$

$$\Rightarrow z = \frac{1-6+5i}{1+9} \qquad (\because i^2 = -1)$$

$$\Rightarrow z = \frac{-5+5i}{10}$$

$$\Rightarrow z = \frac{-1}{2} + \frac{1}{2}i$$

Modulus of the complex number, z

$$= \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}} \qquad \left(\text{If } z = x+iy, \text{ then modulus of } z = \sqrt{x^2+y^2}\right)$$

Ex5.1, 12

Find the multiplicative inverse of the Complex number $\sqrt{5} + 3i$

Multiplicative inverse of z = z^{-1}

Multiplicative inverse of z = $\frac{1}{z}$

Putting $z = \sqrt{5} + 3i$

multiplicative inverse of $\sqrt{5} + 3i = \frac{1}{\sqrt{5} + 3\,i}$

Rationalizing

$$= \frac{1}{\sqrt{5} + 3i} \times \frac{\sqrt{5} - 3i}{\sqrt{5} - 3i}$$

$$= \frac{\sqrt{5} - 3i}{(\sqrt{5} + 3i)(\sqrt{5} - 3i)}$$

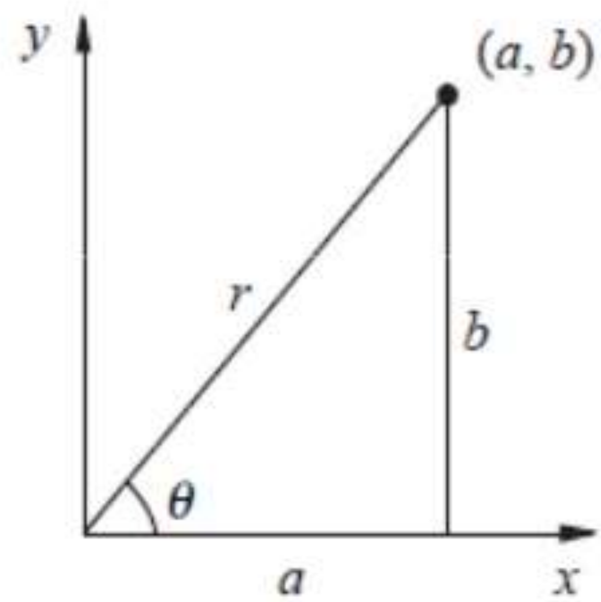
Using $(a - b)(a + b) = a^2 - b^2$

$$= \frac{\sqrt{5} - 3i}{(\sqrt{5})^2 - (3i)^2}$$

$$= \frac{\sqrt{5} - 3i}{5 - 9i^2}$$

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

If $z = a + ib$, is a complex number than in *cartesian form* it is as good as (a, b)



For polar form, let us take $a = r\cos\theta$ and $b = r\sin\theta$

$$\boxed{\begin{aligned} z &= r\cos\theta + i\,r\sin\theta \\ &= r(\cos\theta + i\sin\theta), \\ &= r\,\mathrm{cis}\,\theta \end{aligned}}$$

$r = \sqrt{a^2 + b^2} = |z|$,

$\arg(z) = \mathrm{Arg}(z) + 2\pi k, \quad K = 0, \pm 1, \pm 2 \ldots\ldots\ldots \ldots$

$\mathrm{Arg}(z) = \theta = \tan^{-1}\dfrac{b}{a}, \quad -\pi < \theta \leq \pi$

Let $z = \frac{1+2i}{1-3i}$, then

$$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2+3^2} = \frac{1+5i+6(-1)}{1+9}$$

$$= \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5i}{10} = \frac{-1}{2} + \frac{1}{2}i$$

Let $z = r\cos\theta + ir\sin\theta$

i.e., $r\cos\theta = \frac{-1}{2}$ and $r\sin\theta = \frac{1}{2}$

On squaring and adding, we obtain

$$r^2\left(\cos^2\theta + \sin^2\theta\right) = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$\Rightarrow r^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow r = \frac{1}{\sqrt{2}} \qquad \text{[Conventionally, } r > 0\text{]}$$

$$\therefore \frac{1}{\sqrt{2}}\cos\theta = \frac{-1}{2} \text{ and } \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \frac{-1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad \text{[As } \theta \text{ lies in the II quadrant]}$$

Therefore, the modulus and argument of the given complex number are $\frac{1}{\sqrt{2}}$ and $\frac{3\pi}{4}$ respectively.

Complex Numbers

11) (a) The complex conjugate $3 + 4i$ is $\overline{3 + 4i}$ = ________________

(b) $(3 + 4i)(\overline{3 + 4i})$ = ____________________

12) Consider: $5 - 7i$

(a) What is the real part?

(b) What is the imaginary part?

Write the conjugate of each complex number.

1) $\dfrac{2+\sqrt{-9}}{4}$ __________

2) $5-7i$ __________

4) $7(3+6i)$ __________

5) $\sqrt{11}$ __________

Simplify.

1) i^{73}

2) i^{259}

3) i^{24}

5) i^{543}

6) i^{9}

7) i^{834}

9) i^{910}

10) i^{5}

11) i^{32}

Express the following complex numbers in the standard form. Also find their conjugate: $\frac{1-i}{1+i}$ (ii) $\frac{(1-i)^2}{3-i}$ (iii) $\frac{(2+3i)^2}{2-i}$ (iv) $\frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5-12i}-\sqrt{5-12i}}$

28. What is the conjugate of $\frac{2-i}{(1-2i)^2}$?

29. If $|z_1|=|z_2|$, is it necessary that $z_1 = z_2$?

30. If $\frac{(a^2+1)^2}{2a-i} = x + iy$, what is the value of $x^2 + y^2$?

31. Find z if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.

32. Find $\left|(1+i)\frac{(2+i)}{(3+i)}\right|$

33. Find principal argument of $(1 + i\sqrt{3})^2$.

34. Where does z lie, if $\left|\frac{z-5i}{z+5i}\right| = 1$.